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Application of Progressive Type II Hybrid Censoring Scheme to Estimate Parameters of Kumaraswamy Distribution

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Abstract: This paper considers the Maximum Likelihood Estimators for Kumaraswamy distribution centered on progressive type II hybrid censoring scheme using the expectation maximization algorithm. Kumaraswamy distribution remains of keen consideration in disciplines such as economics, hydrology and survival analysis. To compare the performance of the attained maximum likelihood estimators of Kumaraswamy distribution expectation maximization algorithms is utilized as it is a convenient mechanism in manipulating incomplete data. The presentation of the maximum likelihood estimators via an expectation maximization algorithm is compared using three different amalgamations of censoring schemes. Simulation is utilized to contrast both precision and efficiency. The simulation outcome indicates that there is no notable estimation difference for the three censoring schemes. It also noted that an expectation maximization algorithm has a relatively efficient estimation aimed at Kumaraswamy distribution in progressive type II hybrid censoring scheme. Eventually, an illustration with real life data set is provided and it illustrates how maximum likelihood estimators works in practice under different censoring schemes. It is apparent from the observations made that the estimated values in scheme one is lesser than the other remaining two censoring schemes. It is greater in scheme three than scheme one and scheme two whenever, the three schemes are compared.

Keywords: Kumaraswamy Distribution, Progressive Type II Hybrid Censoring, Maximum Likelihood Estimators, Expectation Maximization Algorithm

1. Introduction

Reliability and life testing trials besides scrutinizing the necessary duration utilized by a unit in an investigation over an interval of time, also consents to the elimination of units from an investigation before failure ensues. In aforesaid fields, usually units stay detached from an investigation with a view to enable them to be suitable for an approximated budget.

Two ordinary categories of censoring techniques, type I and II, are given preference in life testing experiments. Hybrid censoring scheme exists by way of blending both type I and II censoring schemes. Additionally, an intention of initiating Progressive type II censoring scheme is as an end result of type I, type II together with hybrid censoring scheme possessing shortcomings. Hybrid censoring scheme was initiated by Epstein [5] then it turns out to be sought after in 1960 in reliability experiments. Progressive type II censoring scheme has been broadened to progressive type II hybrid censoring scheme by Kundu and Joarder [8]. The study of progressive type II hybrid censoring scheme has been accomplished extensively by Park et al. [15], Lin et al. [10] and Childs et al. [2]. For instance Mokhtari et al. [11] expounded on the inference of progressive type II censored data based on weibull distribution. Yongming and Yimin [18] considered an inference of a distribution known as lomax based on progressive type II hybrid censoring scheme through the use of iterative technique the MLEs are derived. Li and Ma [9] reviewed the inference for a distribution known as the Generalized Rayleigh founded on progressive type II hybrid censoring scheme. In this study, maximum likelihood estimators were obtained using an expectation maximization algorithm.

Kumaraswamy exists as a distribution initiated by Kumaraswamy [7] and could be utilized in numerous expanses such as statistics and hydrology amidst others. The shape parameter of Kumaraswamy distribution has been studied Sultan and Ahmad [16]. It was noted that when the study is based on different priors the posterior standard deviation will tend to decline as the sample size rises. Bayesian estimates that are informative priors performed better compared those that were under non – informative priors. Using progressive type II censored samples, an author known as Gholizadeh et al. [6] concentrated on non-Bayesian as well as Bayesian estimators respectively. The above study took into account the reliability function, shape parameter and failure rate function with consideration given to Kumaraswamy distribution. Whenever comparison using Bayes estimates remained undertaken, an observation was made that maximum likelihood estimators have the least estimated mean squared errors.

Pak et al. [14] considered classical as well as Bayesian estimates of Kumaraswamy distribution which focuses on type II HCS. Bayes estimates provided preferable performances in the study as opposed to the maximum likelihood estimators while centered on informative priors.

Sultana et al. [17] assessed the parameters of Kumaraswamy distribution. The results attained from Tierney and Kadane method were found to be quite good than the other methods. It was noted in this study, that when a comparison was done between Boot-t intervals and asymptotic intervals, Boot-t interval were established to compete relatively well.

Muna, [12] contrasted the dissimilar estimates of the Kumaraswamy distribution. Different methods were compared to estimate the scale parameter as well as the shape parameter. In this study the MLE was the greatest method for sample sizes that are hefty.

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A random variable X is said to be a Kumaraswamy distribution if its probability density function (PDF) and cumulative distribution function (CDF) are respectively given by

$$f(x;\beta,\alpha)=\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1} \tag{1}$$

$$F(x:\beta,\alpha)=1-(1-x^{\alpha})^{\beta} \tag{2}$$

where $0<x<1;\alpha,\beta>0$

The Kumaraswamy distribution remains better suited and can be used in simulation modeling owing to the benefit of a closed form cumulative distribution function.

In this paper, we consider maximum likelihood estimators for Kumaraswamy distribution centred on progressive type II hybrid censoring scheme using an expectation maximization algorithm. Consequently, this paper emphasizes on obtaining the maximum likelihood estimators for Kumaraswamy distribution based on progressive type II hybrid censoring scheme and in addition compares the results with those obtained using real data. Non-identical censoring schemes are employed for the estimation and the comparison of performances of these schemes.

2. Parameter Estimation

2.1. Review of Progressive Type II Hybrid Censoring Scheme

Presume approximately n and independent similar objects stand positioned in a trial at a similar time interval. Likewise, the life times of n objects are symbolized by $X_{1:m:n,}X_{2:m:n},X_{3:m:n}\cdots,X_{m:m:n}$.

An integer m< n of complete failures is normally predetermined by the start of a trial. Predetermined in advance is also time point T, prior to the commencement of the trial. It is observed that $R_1,R_2,R_3,...,R_m$ are m prefixed integers sustaining $R_1+R_2+R_3+...+R_m+m=n$.

During the initial failure's occurrence, $x_{1:m:n}$, R_1 of such residual components is observed to be withdrawn haphazardly. Identically, during the occurrence of a second non-success, $x_{2:m:n},R_2$ of the components that are left are detached.

Whenever the mth non-success occurs, $x_{m:m:n}$, all the $R_m=n-m-R_1-R_2-R_3-...-R_{m-1}-m$ components that have remained alive are withdrawn from the trial and so forth. The trial in Progressive type II hybrid censoring scheme, normally ceases at the duration $x_{m:m:n}$, as soon as an m th failure; $x_{m:m:n}$ transpires prior to the duration T. However, if the mth failure does not take place prior to the duration T yet only J failures transpires prior to the time span T, in an experiment wherever $0\le J\le m$, then towards the duration T, the residual $R_j^*=n-(R_1+R_2+...+R_J)-J$ are withdrawn entirely. Then trial then comes to an end after T.

In PTHCS two cases are obtained as stated below

For Case I: $x_{1:m:n}...x_{m:m:n} if x_{m:m:n}<T$

For Case II: $x_{1:m:n}...x_{J:m:n} if x_{J:m:n}<T<x_{J+1:m:n}$

We note that for case II,

$x_{j:m:n}<T<x_{j+1:m:n}<...<x_{m:m:n}$ and $x_{j+1:m:n},...,x_{m:m:n}$ aren't witnessed.

2.2. MLEs Centered on Progressive Type II Hybrid Censoring Scheme

This segment presents the derivation of MLEs for unknown parameters which include α and β are presented below.

Log likelihood functions for the amalgamated equation is as specified below derived from equation

$$l = D\ln\alpha + D\ln\beta + (\alpha-1)\sum_{i=1}^{D}\ln x_i + (\beta-1)\sum_{i=1}^{D}\ln(1-x_i^{\alpha}) + \beta\sum_{i=1}^{D}R_i\ln(1-x_i^{\alpha}) + \beta R_D^*\ln(1-T^{\alpha}) \quad (3)$$

MLEs for parameters α as well as β may be obtained by differentiating the equation stated above w.r.t α and β as well as equating it to zero

$$\frac{\partial l}{\partial\alpha} = \frac{D}{\alpha} + \sum_{i=1}^{D}\ln x_i - (\beta-1)\sum_{i=1}^{D}\frac{x_i^{\alpha}\ln x_i}{1-x_i^{\alpha}} - \beta\sum_{i=1}^{D}R_i\frac{x_i^{\alpha}\ln x_i}{1-x_i^{\alpha}} - \beta R_D^*\frac{T^{\alpha}\ln T}{1-T^{\alpha}} = 0 \quad (4)$$

$$\frac{\partial l}{\partial\beta} = \frac{D}{\beta} + \sum_{i=1}^{D}\ln(1-x_i^{\alpha}) + \sum_{i=1}^{D}R_i\ln(1-x_i^{\alpha}) + R_D^*\ln(1-T^{\alpha}) = 0 \quad (5)$$

$$\beta^{\wedge} = \frac{-D}{\sum_{i=1}^{D}\ln(1-x_i^{\alpha}) + \sum_{i=1}^{D}R_i\ln(1-x_i^{\alpha}) + R_D^*\ln(1-T^{\alpha})} \quad (6)$$

We can evidently comprehend that the equation (4) has no closed form solution hence the need to adopt the expectation maximization algorithm or Newton Raphson algorithm to obtain the MLEs of α and β.

2.3. Expectation Maximization Algorithm for Kumaraswamy Distribution with Progressive Type II Hybrid Censoring Scheme

EM is an iterative procedure that which was recommended by Dempster et al. [3] and used by Ng et al. [13] has been put forward to facilitate the computation of the maximum likelihood estimators. Denote $z = (z_1, z_2, ..., z_m)$ and $z_j = (z_{j1}, z_{j2}, ..., z_{jR_j})$ where $j = 1,2,3,...m$ be designated as the data that is censored for case I. $z = (z_1, z_2, ..., z_j, z_T)$ with $z_j = (z_{j1}, z_{j2}, ..., z_{jR_j})$ where $j = 1,2,...,J$ and $z_T = (z_{T1}, z_{T2}, ..., z_{TR_j^*})$ be designated as the data that is censored for case II. Censored data is then considered as data that is missing. Let a combination of $(X, Z) = W$ represent a set of data that is complete. The log-likelihood function then may be calculated for both case I as well as case II based on W.

The log-likelihood function for case I is as derived below in (7) and (8).

$$H(w,\alpha,\beta) \propto m\ln\alpha + m\ln\beta + (\alpha-1)\sum_{j=1}^{m}\ln x_j + (\beta-1)\sum_{j=1}^{m}\ln(1-x_j^{\alpha}) + \sum_{j=1}^{m}R_j\ln\alpha + \sum_{j=1}^{m}R_j\ln\beta + (\alpha-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j}\ln z_{jl} + (\beta-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j}\ln(1-z_{jl}^{\alpha}) \quad (7)$$

$$H(w,\alpha,\beta) \propto n\ln\alpha + n\ln\beta + (\alpha-1)\sum_{j=1}^{m}\ln x_j + (\beta-1)\sum_{j=1}^{m}\ln(1-x_j^{\alpha}) + (\alpha-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j}\ln z_{jl} + (\beta-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j}\ln(1-z_{jl}^{\alpha}) \quad (8)$$

For case II

$$H(w,\alpha,\beta) \propto J\ln\alpha + J\ln\beta + (\alpha-1)\sum_{j=1}^{J}\ln x_j + (\beta-1)\sum_{j=1}^{J}\ln(1-x_j^{\alpha}) + \sum_{j=1}^{J}R_j\ln\alpha$$

$$+\sum_{j=1}^{J}R_j\ln\beta + (\alpha-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}\ln z_{jl} + (\beta-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}\ln(1-z_{jl}^{\alpha}) + R_j^*\ln\alpha + R_j^*\ln\beta + (\alpha-1)\sum_{l=1}^{R_j^*}\ln z_{Tl} + (\beta-1)\sum_{l=1}^{R_j^*}\ln(1-z_{Tl}^{\alpha}) \quad (9)$$

$$H(w,\alpha,\beta) \propto n\ln\alpha + n\ln\beta + (\alpha-1)\sum_{j=1}^{J}\ln x_j + (\beta-1)\sum_{j=1}^{J}(1-x_j^{\alpha}) + (\alpha-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}\ln z_{jl} + (\beta-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}\ln(1-z_{jl}^{\alpha})$$

$$+(\alpha-1)\sum_{l=1}^{R_j^*}\ln z_{Tl}+(\beta-1)\sum_{l=1}^{R_j^*}\ln(1-z_{Tl}^{\alpha}) \tag{10}$$

The E-step necessitates the calculation of the pseudo-likelihood component that is attained from $H(w;\alpha,\beta)$ through replacement of whichever function of z_{jl} say h(z_{jl}), by $E(h(z_{jl})/z_{jl}>x_j:m:n)$ and $h(z_{Tl})$ by $E(h(z_{Tl}/z_{Tl}>T)$. Therefore equation (8) and (10) becomes as shown below when the missing is replaced with the conditional expectation.

Consequently, the pseudo-likelihood component for the said two cases is given below;

For case I:

$$H^*(w,\alpha,\beta)\propto n\ln\alpha+n\ln\beta+(\alpha-1)\sum_{j=1}^{m}\ln x_j+(\beta-1)\sum_{j=1}^{m}\ln(1-x_j^{\alpha})$$

$$+(\alpha-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j}E[\ln z_{jl}/z_{jl}>x_{j:m:n}]+(\beta-1)\sum_{j=1}^{m}\sum_{l=1}^{R_j}E[\ln 1-z_{jl}^{\alpha}/z_{jl}>x_{j:m:n}] \tag{11}$$

Case II

$$H^*(w;\alpha,\beta)\propto n\ln\alpha+n\ln\beta+(\alpha-1)\sum_{j=1}^{J}\ln x_j+(\beta-1)\sum_{j=1}^{J}\ln(1-x_j^{\alpha})$$

$$+(\alpha-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}E[\ln(z_{jl}/z_{jl}>x_{j:m:n}]+(\beta-1)\sum_{j=1}^{J}\sum_{l=1}^{R_j}E[\ln(1-z_{jl}^{\alpha})/z_{jl}>x_{j:m:n}]$$

$$+(\alpha-1)\sum_{l=1}^{R_j^*}E[\ln(z_{Tl}/z_{Tl}>T)]+(\beta-1)\sum_{l=1}^{R_J^*}E[\ln(1-z_{Tl})/z_{Tl}>T] \tag{12}$$

To solve the last part in the above equations we introduce the concept of Ng et al. [13].

The M-step will entail the pseudo-likelihood function's maximization by substituting E_1 in equation (11) and E_2 in equation (12) respectively. Suppose that at the k[th] stage, the estimates of (α,β) are $(\alpha^{(k)},\beta^{(k)})$ then $(\alpha^{(k+1)},\beta^{(k+1)})$ for the two possible cases that can be derived are as follows.

Case I:

$$\hat{\alpha}(\beta)=\frac{-n}{\sum_{j=1}^{m}\ln x_j-(\beta-1)\sum_{j=1}^{m}\frac{x_j^{\alpha}\ln x_j}{1-x_j^{\alpha}}+\sum_{j=1}^{m}R_jE_1(x_{j:m:n},\alpha^{(k)},\beta^{(k)})}$$

Case II:

$$\hat{\alpha}(\beta)=\frac{-n}{\sum_{j=1}^{J}\ln x_j-(\beta-1)\sum_{j=1}^{J}\frac{x_j^{\alpha}\ln x_j}{1-x_j^{\alpha}}+\sum_{j=1}^{J}R_jE_1(x_{j:m:n},\alpha^{(k)},\beta^{(k)})+R_j^*E_1(T,\alpha^{(k)},\beta^{(k)})}$$

Therefore the maximization of $H^*(w;\hat{\alpha}(\beta),\beta)$ can be obtained easily by solving.

Case I:

$$\frac{-n}{\beta}=\sum_{j=1}^{m}\ln(1-x_j^{\alpha})+\sum_{j=1}^{m}R_jE_2(x_{j:m:n},\alpha^{(k)},\beta^{(k)})$$

$$\beta^{\wedge} = \frac{-n}{\sum_{j=1}^{m} \ln(1-x_j^{\alpha}) + \sum_{j=1}^{m} R_j E_2(x_{j:m:n}, \alpha^{(k)}, \beta^{(k)})}$$

Case II:

$$\frac{-n}{\beta} = \sum_{j=1}^{J} \ln(1-x_j^{\alpha}) + \sum_{j=1}^{J} R_j E_2(x_{j:m:n}, \alpha^{(k)}, \beta^{(k)}) + R_j^{*} E_2(T, \alpha^{(k)}, \beta^{(k)})$$

$$\beta^{\wedge} = \frac{-n}{\sum_{j=1}^{J} \ln(1-x_j^{\alpha}) + \sum_{j=1}^{J} R_j E_2(x_{j:m:n}, \alpha^{(k)}, \beta^{(k)}) + R_j^{*} E_2(T, \alpha^{(k)}, \beta^{(k)})}$$

Once $\beta^{(k)}$ is obtained, $\alpha^{(k+1)}$ is obtained as $\alpha^{(k+1)}$ = $\alpha^{\wedge} = \alpha^{\wedge}(\beta^{k})$.

3. Simulation Study

We use simulated and actual data to scrutinize performances of the suggested maximum likelihood estimators in this chapter. Aside from the different time points, the precision of the maximum likelihood estimators acquired in different censoring schemes are also assessed. A simulation study is conducted using R statistical software to determine the performance of maximum likelihood estimators using Kumaraswamy distribution under progressive type II hybrid censoring scheme.

3.1. Simulation Algorithm

The values $\alpha = 0.5, \beta = 1.5$ are considered to be the true values that are generated from the parameters of Kumaraswamy distribution in progressive type II hybrid censoring scheme. The three different censoring schemes that are shown below are used with sample sizes of 30, 40 and 60 respectively were used. The m values used are 15, 20, 25, 70 and 80.

The three censoring schemes used are as shown underneath:

One (1): $R_1 = n-m, R_2 = ... = R_m = 0$

Two (2): $R_1 = 0, R_2 = n-m, R_3 = ... = R_m = 0$

Three (3): $R_1 = R_2 = ...R_5 = \left(\frac{n-m}{5}\right), R_6 = R_7 = ... = R_m = 0$

To be able to obtain a distinct outcome for the progressive type II hybrid censoring schemes the time points $T_1 = x_{\frac{m}{3}:m:n} + 0.01$, $T_2 = x_{\frac{m}{2}:m:n}$ and $T_3 = x_{m:m:n} + 1$ respectively are used.

In order to generate a progressive type two hybrid censoring scheme censored samples from Kumaraswamy distribution we utilise the algorithm recommended by Kundu and Joarder, [8] and Balakrishnan and Aggarwala, [1] and that it entails the following steps.

(1) From standard uniform distribution $U[0;1]$ create m independent and identically distributed (i.i.d) random numbers $U_1; U_2; U_3; ...; U_m$

(2) For $i = 1, 2, 3, ..., m$, set $z_i = -\log(1-u_i)$, such that z_i's are independent and identically distributed standard Kumaraswamy distribution variates.

(3) Given n, m and the censoring scheme $R = (R_1, R_2, R_3, ..., R_m)$ attain a type II progressive censored sample $Y_1, Y_2, Y_3, ..., Y_m$ which originates from Kumaraswamy distribution. Let

$$Y_1 = \frac{z_1}{m}$$

$$Y_i = Y_{i-1} + \frac{z_i}{n - \sum_{j=1}^{i-1} R_j - i + 1}$$

(4) For $i = 1, 2, 3, ..., m$ set $W_i = 1 - \exp(-Y_i)$, such that W_i's form a type II progressive censored data from uniform distribution $U[0;1]$.

(5) For $i = 1, 2, 3, ..., m$ set $X_{i:m:n} = F^{-1}(W_i)$

$$F^{-1}(W_i) = [1-(1-W_i)^{\frac{1}{\beta}}]^{\frac{1}{\alpha}}$$

In this paper, the estimations are achieved whenever the obtained absolute difference in the log-likelihood function is below 0.0001. Whenever, we are assessing performance of maximum likelihood estimators, we consider biases and Mean squared errors. For the i^{th} replication of simulated m^{th} algorithm, suppose $\Phi_{mi}^{\wedge}$ is the maximum likelihood estimator of Φ. After simulation, the absolute value of the bias as well as the mean square error are then analysed and remain evaluated as shown below,

$$\text{Bias }(\Phi^{\wedge}) = \frac{1}{h} \left| \sum_{j=1}^{h} (\Phi - \Phi_i^{\wedge} \right|$$

where $\Phi = (\alpha, \beta)$

$$\text{Mean square error } (\Phi^{\wedge}) = \frac{1}{h}\sum_{j=1}^{h}(\Phi - \Phi^{\wedge}{}_{i)})^2$$

R statistical software is used to calculate the biases as well as mean squared errors for different of n, m and T.

3.2. Simulation Results

Table 1. *Mean squared errors and mean biases of the estimators using censoring scheme 1, when* $\alpha = 0.5$ *and* $\beta = 1.5$.

T	N	M	Estimated values		Bias		MSE	
			$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
T_1	30	10	0.5808792	1.1947477	0.0808792	0.3052523	0.0065414	0.0931790
		15	0.4777726	1.3249943	0.0222274	0.1750057	0.0004941	0.0306270
	40	15	0.5639237	1.3054146	0.0639237	0.1945854	0.0040862	0.0378635
		20	0.4665673	1.3732688	0.0334327	0.1267312	0.0011177	0.0160608
	60	25	0.5212912	1.3361205	0.0212912	0.1638795	0.0004533	0.0268565
		30	0.4779191	1.5424446	0.0220809	0.0424446	0.0004876	0.0018015
T_2	30	10	0.5821163	1.3319922	0.0821163	0.1680078	0.0067431	0.0282266
		15	0.4930337	1.4654171	0.0069663	0.0345829	4.852934e-05	0.0011960
	40	15	0.5611578	1.4048896	0.0611578	0.0951104	0.0037403	0.0090460
		20	0.4891187	1.5826262	0.0108813	0.0826262	0.0001184	0.0068271
	60	25	0.5266963	1.4684930	0.0266963	0.0315070	0.0007127	0.0009927
		30	0.4933643	1.7130167	0.0066357	0.2130167	4.403251e-05	0.0453761
T_3	30	10	0.4649507	1.176986	0.0350493	0.3230140	0.0350493	0.3230140
		15	0.4768134	1.480034	0.0231866	0.0199660	0.0231866	0.0199660
	40	15	0.5035833	1.239549	0.0035833	0.2604510	0.0035833	0.2604510
		20	0.4815965	1.528928	0.0184035	0.0289280	0.0184035	0.0289280
	60	25	0.4556201	1.320640	0.0443799	0.1793600	0.0443799	0.1793600
		30	0.5329161	1.541242	0.0329161	0.0412420	0.0329161	0.0412420

Table 2. *Mean squared errors and mean biases of the estimators using censoring scheme 2, when* $\alpha = 0.5$ *and* $\beta = 1.5$.

T	N	M	Estimated values		Bias		MSE	
			$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
T_1	30	10	0.6135017	1.3319922	0.1135017	0.1627172	0.0128826	0.0282266
		15	0.5050327	1.4654171	0.0199216	0.0762419	0.0003969	0.0058128
	40	15	0.5896615	1.4048896	0.0896615	0.0951104	0.0080392	0.0090460
		20	0.4438913	1.5228429	0.0561087	0.0228429	0.0031482	0.0005218
	60	25	0.5434327	1.4648684	0.0434327	0.0315316	0.0018864	0.0012342
		30	0.4857485	1.5297197	0.0142515	0.0297197	0.0002031	0.0008833
T_2	30	10	0.5821163	1.3319922	0.0821163	0.1680078	0.0067431	0.0282266
		15	0.4693789	1.3554794	0.0306211	0.1445206	0.0009377	0.0208862
	40	15	0.5611578	1.4048896	0.0611578	0.0951104	0.0037403	0.0090460
		20	0.4891187	1.5826262	0.0108813	0.0826262	0.0001184	0.0068271
	60	25	0.5266963	1.4684930	0.0266963	0.0315070	0.0007127	0.0009927
		30	0.4998374	1.4986043	0.0001626	0.0013957	2.643876e-08	1.947978e-06
T_3	30	10	0.4649507	1.1769860	0.0350493	0.3230140	0.0350493	0.3230140
		15	0.4829745	1.5568400	0.0170255	0.0568400	0.0002899	0.0032308
	40	15	0.5334452	1.4752150	0.0334452	0.0247850	0.0011186	0.0006143
		20	0.4815965	1.5289280	0.0184035	0.0289280	0.0184035	0.0289280
	60	25	0.5152384	1.5337420	0.0152384	0.0337420	0.0002322	0.0011385
		30	0.5012384	1.5004010	0.0012384	0.0004010	1.533635e-06	1.60801e-07

Table 3. *Mean squared errors and mean biases of the estimators using censoring scheme 3, when* $\alpha = 0.5$ *and* $\beta = 1.5$.

T	N	M	Estimated values		Bias		MSE	
			$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
T_1	30	10	0.4739442	1.1216879	0.0260558	0.3783121	0.0006789	0.1431200
		15	0.5117597	1.5802887	0.0117597	0.0802887	0.0001383	0.0064463
	40	15	0.5741457	1.2909942	0.0741457	0.2090058	0.0054976	0.0436834
		20	0.4889018	1.5228594	0.0110982	0.0228594	0.0001232	0.0005226
	60	25	0.50557010	1.4868295	0.0055701	0.0131705	3.102601e-05	0.0001735
		30	0.4979142	1.5090407	0.0020858	0.0090407	4.350562e-06	8.173426e-05

T	N	M	Estimated values		Bias		MSE	
			$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
T_2	30	10	0.5422267	1.3933570	0.0422267	0.1066430	0.0017831	0.0113727
		15	0.4693694	1.4720373	0.0306306	0.0279627	0.0009382	0.0007819
	40	15	0.5440085	1.5402768	0.0440085	0.0402768	0.0019367	0.0016222
		20	0.4616332	1.4682133	0.0383668	0.0317867	0.0014720	0.0010104
	60	25	0.5169319	1.5315904	0.0169319	0.0315904	0.0002867	0.0009980
		30	0.4979463	1.4909888	0.0020537	0.0090112	4.217684e-06	8.120173e-05
T_3	30	10	0.5389908	1.3602650	0.0389908	0.1397350	0.0015203	0.0195259
		15	0.4527546	1.4780510	0.0472454	0.0219490	0.0022321	0.0004818
	40	15	0.5608686	1.2493020	0.0608686	0.0250698	0.0037050	0.0628495
		20	0.5155527	1.4819340	0.0155527	0.0180660	0.0002419	0.0003264
	60	25	0.5109621	1.4832370	0.0109621	0.0167630	0.0001202	0.0002810
		30	0.4957498	1.5015300	0.0042502	0.0015300	1.80642e-05	2.3409e-06

A summary of results from Tables 1-3 is provided below: -

From table 1, 2 and 3, it is noted that results above, are generated using three censoring schemes namely, scheme one (1), scheme two (2) and scheme three (3) respectively. It is witnessed that an expectation maximization algorithm has a relatively efficient estimation meant for Kumaraswamy distribution under progressive type II hybrid censoring scheme. We also note the following:

1) For fixed sample sizes of n and time interval T, the biases and mean square errors are observed to be decreasing for most of the estimated parameters as also the number of witnessed failures, m increases.
2) For a fixed time point T and fixed amount of witnessed non-successes, m, equally sample sizes n keeps increasing, biases and mean squared errors have been observed to increase for majority of the estimates.
3) For specified number of witnessed non-successes m as well as sample sizes n, as the trial's pre-determined time point increases, biases as well as mean squared errors for majority of the estimated parameters are observed to be decreasing as expected.
4) At fixed time point T and sample sizes n, as number of witnessed failures m increases most of the projected values of α and β tend to give smaller values which appear to converge more rapidly to the true values of α and β.
5) It is also important to take note that generally, no much significant estimation differences for the three censoring schemes for fixed time point T, number of observed failures m as well as sample sizes n.

3.3. Numerical Analysis

To make evident the application and utilization of the proposed methods to real data a simulation study is undertaken with a purpose of contrasting the performance. Actual data similar to the data set used by El- Sagheer, [4] is utilized. The data set is acquired from the reservoir of Shasta located in California, USA. The monthly capacity statistics were availed from February 1991 to 2010. The data were converted to the interval [0, 1] by El- Sagheer [4], to ensure that the converted data follow Kumaraswamy distribution. Real data aids to illustrate how maximum likelihood estimator using expectation maximization algorithm works in practise.

The maximum capacity of the reservoir was observed to be 4,552,000, El-Sagheer, [4] and it was established that the Kumaraswamy distribution fits and works relatively fine for the capacity data.

Table 4. *Data set of monthly capacity statistics from the reservoir of Shasta.*

YEAR	Percentage of total capacity	Capacity	YEAR	Percentage of total capacity	Capacity
1991	0.338936	1,542,838	2001	0.768007	3,495,969
1992	0.431915	1,966,077	2002	0.843485	3,839,544
1993	0.759932	3,456,209	2003	0.787408	3,584,283
1994	0.724626	3,298,496	2004	0.849868	3,834,600
1995	0.757583	3,448,519	2005	0.695970	3,168,056
1996	0.811556	3,694,201	2006	0.842316	3,834,224
1997	0.785339	3,574,861	2007	0.828689	3,772,193
1998	0.783660	3,567,220	2008	0.580194	2,641,041
1999	0.815627	3,712,733	2009	0.430681	1,960,458
2000	0.847413	3,857,423	2010	0.742563	3,380,147

We consider the progressive type II hybrid censored samples of size m=10 and m=12 of the proportions of total capacity generated randomly from n=20 observations. The schemes used are as indicated earlier.

The maximum likelihood estimates below were established via an expectation maximization algorithm. Based on sample data, the outcomes are generated in tables 5, 6 and 7 below.

Table 5. *Progressive type II hybrid censored data from Kumaraswamy distribution with a sample size of 20 when m=10 and 12, is generated under scheme 1.*

T	N	M	Estimated values	
			$\hat{\alpha}$	$\hat{\beta}$
T_1	20	10	0.1961361	2.343943
		12	0.2946307	2.649166
T_2	20	10	0.2401888	2.339306
		12	0.3691621	2.664499
T_3	20	10	0.2425485	2.400338
		12	0.3452707	2.711777

Table 6. *Progressive type II hybrid censored data from Kumaraswamy distribution with a sample size of 20 when m=10 and 12, is generated under scheme 2.*

T	N	M	Estimated values	
			$\hat{\alpha}$	$\hat{\beta}$
T_1	20	10	0.2417554	2.717959
		12	0.4309088	3.053647
T_2	20	10	0.2427502	2.727710
		12	0.4084040	3.162329
T_3	20	10	0.2458876	2.763019
		12	0.4065904	3.145414

Table 7. *Progressive type II hybrid censored data from Kumaraswamy distribution with a sample size of 20 when m=10 and 12, is generated under scheme 3.*

T	N	M	Estimated values	
			$\hat{\alpha}$	$\hat{\beta}$
T_1	20	10	0.2559082	2.727710
		12	0.4048950	3.145414
T_2	20	10	0.2866438	2.763019
		12	0.4046776	3.154011
T_3	20	10	0.2661871	2.655114
		12	0.4759144	3.053647

When comparing the estimated values of $\alpha^{\wedge}$ and $\beta^{\wedge}$ as in the three different censoring schemes generated by tables 5, 6 and 7, we observed that the estimated values in the first censoring scheme is lesser than the other remaining two censoring schemes. It is greater in the third scheme than the first scheme and the second scheme.

4. Conclusion

In this paper, the challenge of estimating maximum likelihood estimators for two parameter Kumaraswamy distribution using progressive type II hybrid censoring scheme was tackled. The maximum likelihood estimators were attained using expectation maximization algorithm. The simulation results of biases and mean squared errors yielded the following observations for all the three censoring scheme.

1) For fixed sample sizes of n as well as time interval T, just as even the quantity of witnessed non-successes, m increases, the biases and mean square errors are observed to be decreasing for most of the estimated parameters.
2) For a fixed duration T and set number of witnessed no-successes, m, as the sample sizes n, rises, biases and mean squared errors is indeed observed to be rising for majority of the estimates.
3) For set number of witnessed failures m and sample sizes n, as the pre-determined duration of the trial rises, the biases and mean squared errors for most such estimated parameters are observed to be decreasing as expected.
4) At fixed time point T and sample sizes n, as number of witnessed failures m increases most of the projected values of α and β tend to give smaller values which appear to converge more rapidly to the true values of α and β.

In addition, a real data analysis was also carried out and an observation has been made, that when comparing the estimated values of and in the three different schemes were used. We also observed that the estimated values in the first censoring scheme are lesser than the other remaining two censoring schemes. It is greater in the third scheme than the first scheme and the second scheme.

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